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Note on the method of matched-asymptotic expansions for determining the force acting on a particle

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This paper is an addendum to the article by Candelier, Mehaddi & Vauquelin (2013) where the motion of a particle in a stratified fluid is investigated theoretically, at small Reynolds and Péclet numbers. We review briefly the method of matched asymptotic expansions which is generally used in order to determine the force acting on a particle embedded in a given flow, in order to account for small, but finite, inertia effects. As part of this method, we present an alternative matching procedure, which is based on a series expansion of the far-field solution of the problem, performed in the sense of generalized functions. The way to perform such a series is presented succinctly and a simple example is provided.

1. Introduction

The prediction of the trajectories followed by small spherical inclusions embedded in a given flow has been, and remains, a topic of intense interest. This problem indeed has obvious applications in many fields of physics, as well as in engineering science. In order to determine accurately these trajectories, a detailed understanding of the force acting on the particles is required.

In the case where the non-linear convective terms, involved in the Navier-Stokes equations written in a frame of reference moving with the particle, can be totally neglected, the force acting on an isolated solid sphere immersed in a Newtonian fluid is provided by the well-known Basset-Boussinesq-Oseen (BBO) equation (Boussinesq 1885; Basset 1888). The various terms appearing in this equation are the drag, the added-mass force and the history force. An equivalent equation for fluid inclusions immersed in an unsteady uniform flow (in the absence of inertia effects) has been obtained by Gorodtsov (1975) (see also Yang & Leal 1991; Galindo & Gerbeth 1993).

However, the creeping-flow assumption is no longer valid in many situations. For example, when an inclusion is released into a quiescent fluid, the unsteadiness of the velocity perturbation eventually vanishes while convective terms are no longer negligible far from the inclusion (Oseen's problem). The determination of the particle-induced flow therefore requires us to solve a steady equation, in which linearised convective terms are involved (see Proudman & Pearson 1957). As a result, it is found that the drag on the sphere is enhanced. Another striking example is provided by the lift force which appears when a particle is embedded in a pure shear flow, owing to convective inertia effects (Saffman 1965, 1968; McLaughlin 1991; Asmolov & McLaughlin 1999; Candelier & Souhar 2007), as well as a shear-induced drag correction (see Harper & Chang 1968; Miyazaki, Bedeaux & Bonet Avalos 1995). Similarly, when particles are immersed in a solid-body rotation

flow, convective inertia effects induce lift forces, as well as drag corrections in both the axial and the radial directions (Childress 1964; Herron, Davis & Bretherton 1975; Gotoh 1990; Candelier, Angilella & Souhar 2005; Candelier & Angilella 2006; Candelier 2008).

In order to determine the influence of inertia terms on the force acting on a particle, the method of matched asymptotic expansions is generally used, and more specifically, that devised by Childress (1964) (see also Saffman 1965) and generalized to fluid inclusion by Legendre & Magnaudet (1997). In this paper, the classical method is first presented in a general way, and then an alternative matching procedure, which is based on a series expansion of the far-field solution of the problem, performed in the sense of generalized functions, is proposed.

2. Description of the classical method

Formally, in the problems mentioned in the introduction, the non-dimensional perturbed fluid motion equations, in which, in particular, lengths are scaled by the radius of the particle, can be written as follows

$$\nabla \cdot \mathbf{w} = 0, \quad (2.1)$$

$$-\nabla p + \Delta \mathbf{w} = \mathbf{q}_\epsilon, \quad (2.2)$$

$$\mathbf{w} = \mathbf{u}_r \text{ on } r = 1, \quad \text{and} \quad \mathbf{w} \rightarrow 0 \text{ as } r \rightarrow \infty, \quad (2.3)$$

where \mathbf{u}_r is the relative velocity of the particle (i.e. $\mathbf{u} - \mathbf{v}$, where \mathbf{u} is the velocity of the particle, \mathbf{v} , that of the fluid, and where the rotation of the particle has been neglected). In this kind of problem, the creeping flow equations are perturbed by a term \mathbf{q}_ϵ , which is such that

$$\lim_{\epsilon \rightarrow 0} \mathbf{q}_\epsilon \rightarrow 0, \quad (2.4)$$

and whose analytical expression naturally depends on the case considered.

In a region close to the particle, i.e. characterized by $r \sim 1$, and which is usually called the inner zone, the solution of equations (2.1) to (2.3) is expanded formally as

$$\mathbf{w} = \mathbf{w}_0 + \epsilon \mathbf{w}_1 + O(\epsilon^2) \quad \text{and} \quad p = p_0 + \epsilon p_1 + O(\epsilon^2) \quad (2.5)$$

where the zeroth-order velocity and pressure satisfy the creeping flow equations (i.e. $\mathbf{q}_\epsilon = 0$). As shown by Saffman (1965), the inner problem is generally not regular since the boundary condition at infinity cannot be satisfied by the term \mathbf{w}_1 which is found to be of order $O(r)$ for large r in the great majority of cases. As a consequence, the first-order correction terms have to be matched to an outer solution, i.e. a solution which is valid far from particle (i.e. $r \gg 1$).

The far-field solution is obtained by considering that in this region, the inclusion is seen by the fluid as a punctual force, modelled by Dirac-source term whose strength corresponds to that of a Stokes drag (with a minus sign), and which leads us to

$$-\nabla p + \Delta \mathbf{w} + 6\pi \mathbf{u}_r \delta(\mathbf{x}) = \mathbf{q}_\epsilon. \quad (2.6)$$

In terms of stretched coordinates $\tilde{\mathbf{x}} = \epsilon \mathbf{x}$, and after noticing that $\delta(\mathbf{x}) = \epsilon^3 \delta(\tilde{\mathbf{x}})$, equation (2.6) can be re-written as follows

$$-\tilde{\nabla} p' + \tilde{\Delta} \mathbf{w}' + 6\pi \mathbf{u}_r \epsilon \delta(\tilde{\mathbf{x}}) = \mathbf{q}'_1, \quad (2.7)$$

where the fluid velocity, the pressure and the perturbation term, are now denoted with a prime, in order to distinguish them from the variables written in normal coordinates.

Let us denote by $\epsilon \mathbf{w}'_{\text{out}}$ the (normalized) solution of equation (2.7). For later convenience, let us also introduce $\epsilon \mathbf{w}'_{St}$, the well-known Stokeslet solution (here written in

terms of stretched-coordinates) which simply corresponds to the solution of (2.7) in the particular case $\mathbf{q}'_1 = 0$.

The last step of the method consists in matching the inner and the outer solution in a region where both the solutions are supposed to be valid. Such a region is actually defined by $r \sim 1/\epsilon$, where we should have

$$\mathbf{w}_0 + \epsilon \mathbf{w}_1 \sim \epsilon \mathbf{w}'_{\text{out}}. \quad (2.8)$$

In view of the fact that in the matching region $r \gg 1$, the velocity \mathbf{w}_0 naturally tends to the Stokeslet solution, i.e. $\mathbf{w}_0 \rightarrow \epsilon \mathbf{w}'_{St}$, it can be inferred, after simplifying the parameter ϵ that

$$\mathbf{w}_1(\mathbf{x}) \sim \mathbf{w}'_{\text{out}}(\tilde{\mathbf{x}}) - \mathbf{w}'_{St}(\tilde{\mathbf{x}}).$$

After rewriting $\mathbf{w}_1(\mathbf{x})$ and $\mathbf{w}'(\tilde{\mathbf{x}}) - \mathbf{w}'_{St}(\tilde{\mathbf{x}})$ in terms of an intermediary variable (see for instance Hinch 1991)

$$\boldsymbol{\eta} = \frac{\tilde{\mathbf{x}}}{\epsilon^{1-\alpha}} = \epsilon^\alpha \mathbf{x}, \quad \text{where } 0 < \alpha < 1,$$

and by taking the limit when $\epsilon \rightarrow 0$ for a fixed value of $\boldsymbol{\eta}$, we are led to the following matching condition

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{w}_1(\mathbf{x}) = \lim_{|\tilde{\mathbf{x}}| \rightarrow 0} (\mathbf{w}'_{\text{out}}(\tilde{\mathbf{x}}) - \mathbf{w}'_{St}(\tilde{\mathbf{x}})). \quad (2.9)$$

Note that in general, the solution of (2.7) is obtained by using Fourier transforms, in order to deal with the Dirac-source term involved in it. Thus, by defining the Fourier transform as follows

$$\hat{\mathbf{f}}(\mathbf{k}) = \int_{\mathbb{R}^3} \mathbf{f}(\tilde{\mathbf{x}}) \exp(-i \mathbf{k} \cdot \tilde{\mathbf{x}}) d\tilde{\mathbf{x}}, \quad (2.10)$$

where i is the imaginary unit (i.e. $i^2 = -1$), and the inverse Fourier transform by

$$\mathbf{f}(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{\mathbf{f}}(\mathbf{k}) \exp(i \mathbf{k} \cdot \tilde{\mathbf{x}}) d\mathbf{k},$$

equation (2.9) generally reads as

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{w}_1(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k}) - \hat{\mathbf{w}}'_{St}(\mathbf{k})) d\mathbf{k}. \quad (2.11)$$

Physically, (2.11) means that in the matching region, the perturbation term \mathbf{w}_1 matches a uniform stream of velocity. In practice, this uniform flow is linked to the relative velocity of the particle by a linear relation of the form

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k}) - \hat{\mathbf{w}}'_{St}(\mathbf{k})) d\mathbf{k} = -\mathbf{M} \cdot \mathbf{u}_r$$

where \mathbf{M} defines a mobility-like tensor. In return, this outer uniform flow exerts on the particle, an additional Stokes drag so that the force acting on the particle finally reads as

$$\mathbf{f}_1 = -6\pi(\mathbf{I} + \epsilon \mathbf{M}) \cdot \mathbf{u}_r,$$

where \mathbf{I} is the identity tensor.

As discussed in the introduction of the paper, the method of matched asymptotic expansions has been applied successfully in many physical situations. However, in some cases, the integral involved in the matching procedure (2.11) may be difficult to solve, owing to the complexity of the analytical expression of its integrand. In what follows,

an alternative matching procedure is proposed, which can help to perform the matching between the inner and the outer expansions in the cases where the classical procedure fails.

3. Series expansions of generalized functions

The alternative matching procedure is based on series expansions of generalized functions. Let us then first recall basic definitions and fundamental results concerning distributions.

Suppose that $f(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a locally integrable function, and let $\phi(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a test function in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ (function space of all infinitely differentiable functions that are rapidly decreasing at infinity). The tempered distribution \mathcal{T}_f which corresponds to the function f is defined by

$$\langle \mathcal{T}_f, \phi \rangle = \int_{\mathbb{R}^3} f(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}.$$

According to this definition, several other mathematical tools can be defined, as for instance, the partial derivative of a distribution with respect to a spatial coordinates, say x_i ,

$$\left\langle \frac{\partial \mathcal{T}_f}{\partial x_i}, \phi \right\rangle \triangleq - \left\langle \mathcal{T}_f, \frac{\partial \phi}{\partial x_i} \right\rangle. \quad (3.1)$$

Also, the Fourier transform of a distribution, that we shall denote by $\mathcal{F}(\mathcal{T}) = \hat{\mathcal{T}}$ can be defined by

$$\left\langle \hat{\mathcal{T}}, \phi \right\rangle \triangleq \left\langle \mathcal{T}, \hat{\phi} \right\rangle. \quad (3.2)$$

Note that the symbol \triangleq used in these two last equations stands for 'equal by definition'.

Now in the case where the function $f(\mathbf{x})$ is integrable over the whole space, i.e.

$$\int_{\mathbb{R}^3} f(\mathbf{x}) d\mathbf{x} = C, \quad \text{where } C \text{ is a constant} \quad (3.3)$$

a fundamental result is that, in the sense of generalized function,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} f\left(\frac{\mathbf{x}}{\epsilon}\right) \rightarrow C \delta(\mathbf{x}). \quad (3.4)$$

The demonstration of this result is rather simple and it is instructive to examine it (see for instance Boccara 1997). Briefly, to demonstrate (3.4), stretched coordinates $\tilde{\mathbf{x}} = \epsilon \mathbf{x}$ are generally introduced, which allows the function

$$g_\epsilon(\tilde{\mathbf{x}}) = \frac{1}{\epsilon^3} f\left(\frac{\tilde{\mathbf{x}}}{\epsilon}\right),$$

to be defined. By a change of variable in (3.3), one can readily verify that,

$$\int_{\mathbb{R}^3} g_\epsilon d\tilde{\mathbf{x}} = C, \quad (3.5)$$

so that the effect of the distribution \mathcal{T}_{g_ϵ} on a test function ϕ can be arbitrarily re-written as

$$\langle \mathcal{T}_{g_\epsilon}, \phi \rangle = \int_{\mathbb{R}^3} g_\epsilon(\tilde{\mathbf{x}}) (\phi(\tilde{\mathbf{x}}) - \phi(0)) d\tilde{\mathbf{x}} + C\phi(0).$$

Taking the limit when $\epsilon \rightarrow 0$, and re-introducing unstretched coordinates therefore yields

$$\lim_{\epsilon \rightarrow 0} \langle T_{g_\epsilon}, \phi \rangle = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(\mathbf{x}) \left(\phi(\tilde{\mathbf{x}}) - \phi(0) \right) d\mathbf{x} + C\phi(0).$$

For a fixed value of \mathbf{x} , and because f is locally integrable, the integral term vanishes when $\epsilon \rightarrow 0$, so that

$$\lim_{\epsilon \rightarrow 0} \langle T_{g_\epsilon}, \phi \rangle = C\phi(0) \iff \lim_{\epsilon \rightarrow 0} T_{g_\epsilon} = C\delta.$$

To take the analyse one step further, let us now consider a generalized function, say \mathcal{T} , which is defined by

$$\mathcal{T} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{3+k}} f\left(\frac{\mathbf{x}}{\epsilon}\right), \quad (3.6)$$

where k is an arbitrary integer. Let us further consider three other integers, ℓ , m and n , which are such that $\ell + m + n = k$. According to (3.4), it can be inferred that, for any combination of ℓ , m and n , we should have

$$x_1^\ell x_2^m x_3^n \mathcal{T} = C \delta(\mathbf{x}), \quad \text{where} \quad C = \int_{\mathbb{R}^3} x_1^\ell x_2^m x_3^n f(\mathbf{x}) d\mathbf{x}. \quad (3.7)$$

In terms of generalized function, solving equation (3.7) leads us to

$$\mathcal{T} = C \frac{(-1)^k}{\ell!m!n!} \frac{\partial^k \delta}{\partial x_1^\ell \partial x_2^m \partial x_3^n}. \quad (3.8)$$

Note that any derivative of the delta distribution of lower order than k is also a solution of (3.7), however such terms are not compatible with (3.6) so that they should be zero.

By taking into account every possible combination, we are finally led to the following results:

- in the case $k = 1$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^4} f\left(\frac{\mathbf{x}}{\epsilon}\right) = \sum_{i=1}^3 C_i \frac{\partial \delta}{\partial x_i} \quad \text{with} \quad C_i = \int_{\mathbb{R}^3} x_i f(\mathbf{x}) d\mathbf{x},$$

- in the case $k = 2$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^4} f\left(\frac{\mathbf{x}}{\epsilon}\right) = \sum_{i=1}^3 \sum_{j \geq i}^3 C_{ij} \frac{\partial^2 \delta}{\partial x_i \partial x_j},$$

$$\text{with } C_{ij} = \int_{\mathbb{R}^3} x_i x_j f(\mathbf{x}) d\mathbf{x} \text{ if } i \neq j, \text{ and else } C_{ii} = \frac{1}{2} \int_{\mathbb{R}^3} x_i^2 f(\mathbf{x}) d\mathbf{x},$$

- and so on

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{3+k}} f\left(\frac{\mathbf{x}}{\epsilon}\right) \rightarrow \underbrace{\sum_{i_1=1}^3 \sum_{i_2 \geq i_1}^3 \cdots \sum_{i_n \geq i_{n-1}}^3}_{k \text{ times}} C_{i_1 i_2 \dots i_n} \frac{\partial^k \delta(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}},$$

$$\text{with } C_{i_1 i_2 \dots i_n} = \frac{(-1)^k}{\ell!m!n!} \int_{\mathbb{R}^3} x_1^\ell x_2^m x_3^n f(\mathbf{x}) d\mathbf{x},$$

where ℓ , m and n are determined by the number of occurrences, respectively, of the indices 1, 2 and 3 in the sequence $i_1 i_2 \dots i_n$.

3.1. A simple example

In order to illustrate how such results can be used to approximate a perturbed form of the Green function of a differential equation, let us first consider a very simple example based on a steady Schrödinger-like equation of the form :

$$\Delta f - \epsilon^2 f = \delta .$$

To determine the Green function of this equation, the (spatial) Fourier transform defined in (2.10) can be used, which yields

$$\hat{f} = -\frac{1}{k^2 + \epsilon^2} , \quad (3.9)$$

and then, calculating the inverse Fourier transform of \hat{f} leads us to

$$\mathcal{F}^{-1}(\hat{f}) = f(\mathbf{x}) = -\frac{\exp(-\epsilon r)}{4\pi r} \quad \text{where } r = |\mathbf{x}| .$$

This solution can be expanded, with respect to ϵ , as follows

$$-\frac{\exp(-\epsilon r)}{4\pi r} = -\frac{1}{4\pi r} + \frac{\epsilon}{4\pi} - \frac{r\epsilon^2}{8\pi} + \frac{r^2\epsilon^3}{24\pi} + O(\epsilon^4) . \quad (3.10)$$

Obviously, such a series could not have been retrieved directly from a naive expansion of (3.9) (i.e. performed in the sense of classical function) since only powers of 2 would be involved in it. In contrast, (3.10) can be retrieved if the series is performed in the sense of generalized function. Similarly as for classical functions, such a series reads as

$$-\frac{1}{k^2 + \epsilon^2} = \hat{\mathcal{T}}_0 + \epsilon \hat{\mathcal{T}}_1 + \epsilon^2 \hat{\mathcal{T}}_2 + \dots \epsilon^n \hat{\mathcal{T}}_n + O(\epsilon^{n+1})$$

where

$$\mathcal{T}_n = \frac{1}{n!} \lim_{\epsilon \rightarrow 0} \frac{d^n}{d\epsilon^n} \left(-\frac{1}{k^2 + \epsilon^2} \right) . \quad (3.11)$$

In our simple example, by calculating the first term, we are led to

$$\hat{\mathcal{T}}_0 = \lim_{\epsilon \rightarrow 0} -\frac{1}{k^2 + \epsilon^2} = -\frac{1}{k^2} \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\mathcal{T}}_0) = -\frac{1}{4\pi r} .$$

For the second term,

$$\hat{\mathcal{T}}_1 = \lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} \left(-\frac{1}{k^2 + \epsilon^2} \right) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} \frac{2}{((k/\epsilon)^2 + 1)^2} .$$

According to the fact that

$$\int_{\mathbb{R}^3} \frac{2}{((k^2 + 1)^2)} d\mathbf{k} = 2\pi^2 ,$$

this yields

$$\hat{\mathcal{T}}_1 = 2\pi^2 \delta(\mathbf{k}) \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\mathcal{T}}_1) = \frac{1}{4\pi} ,$$

since $\mathcal{F}^{-1}(\delta) = 1/(8\pi^3)$. By pursuing the expansion in a similar way, we are finally led to

$$\begin{aligned} \hat{\mathcal{T}}_2 &= \frac{1}{k^4} \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\mathcal{T}}_2) = -\frac{r}{8\pi} , \\ \hat{\mathcal{T}}_3 &= -\frac{\pi^2}{3} \Delta_k \delta(\mathbf{k}) \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\mathcal{T}}_3) = \frac{r^2}{24\pi} , \quad \text{etc} \dots \end{aligned}$$

so that (3.10) is indeed recovered.

4. The alternative matching procedure

According to the results presented in the previous section, an alternative matching procedure can now be proposed. Indeed, by considering the fact that the parameter ϵ is small compared to unity, the Fourier transform of the solution of the (unstretched) outer equation (2.6) can be expanded in terms of generalized functions

$$\hat{\mathbf{w}} = \hat{\mathcal{T}}_0 + \epsilon \hat{\mathcal{T}}_1 + \epsilon^2 \hat{\mathcal{T}}_2 + \dots + \epsilon^n \hat{\mathcal{T}}_n$$

where similarly as in the previous case, the generalized functions \mathcal{T}_n are determined by

$$\mathcal{T}_n = \frac{1}{n!} \lim_{\epsilon \rightarrow 0} \frac{d^n \hat{\mathbf{w}}}{d\epsilon^n}. \quad (4.1)$$

According to (2.4), it is readily found that $\hat{\mathcal{T}}_0$ simply corresponds to the Fourier transform of a Stokeslet, so that, in the matching zone, its inverse (spatial) Fourier transform naturally matches the leading order term \mathbf{w}_0 of the inner expansion (2.5).

In our problems, the second term is always found to be of the form

$$\hat{\mathcal{T}}_1 = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} \mathbf{f} \left(\frac{\mathbf{k}}{\epsilon} \right),$$

which implies that

$$\hat{\mathcal{T}}_1 = \mathbf{C} \delta(\mathbf{k}) \quad \text{where} \quad \mathbf{C} = \int_{\mathbb{R}^3} \left. \frac{d\hat{\mathbf{w}}}{d\epsilon} \right|_{\epsilon=1} d\mathbf{k}.$$

By noticing that the inverse spatial Fourier transform of $\delta(\mathbf{k})$ is given by $1/(2\pi)^3$, and similarly as in the classical method, is observed that in the matching zone $r \sim 1/\epsilon$, the perturbation term \mathbf{w}_1 of the inner expansion (2.5) matches a uniform velocity stream given by $\mathbf{C}/(2\pi)^3$, and we are led to the same conclusions as for the classical matching procedure.

5. Concluding remarks

It is worth mentioning that formally

$$\frac{1}{8\pi^3} \int_{\mathbb{R}^3} \left. \frac{d\hat{\mathbf{w}}}{d\epsilon} \right|_{\epsilon=1} d\mathbf{k} = \int_{\mathbb{R}^3} (\hat{\mathbf{w}}'_{\text{out}}(\mathbf{k}) - \hat{\mathbf{w}}'_{St}(\mathbf{k})) d\mathbf{k}$$

which means that the two matching procedures obviously provide us with the same result. In some ways, the difference between the two approaches can be viewed as an inversion between taking the limit when $\epsilon \rightarrow 0$, and then performing the integration, or conversely, performing first the integration, and then taking the limit. Also, let us mention that the alternative matching procedure proposed here has been tested in many configurations where the classical method applies well as, in particular, the problem considered by Oseen (Proudman & Person 1957) or that considered by Herron *et al.* (1975).

Let us finally mention that this alternative method has been specifically developed to allow us to determine the drag correction induced by the flow perturbation on a particle in the problem recently addressed by Ardekani & Stoker (2010). These authors have investigated the flow produced by a point force, intended to represent a settling particle, in a stratified fluid, at small Reynolds and Péclet numbers. In particular, in this study, the creeping flow solution is perturbed by buoyancy effects, and therefore, it can be cast into the formalism described in §2, except that the integral involved in the classical matching procedure (2.11) cannot be solved analytically, owing to the complexity of the analytical expression of its integrand. These results, which have also been generalized to

the unsteady case, have been the subject of a companion paper by Candelier, Mehaddi & Vauquelin 2013.

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